WP. No. : SAUFE-WP-2017-003

Exchange rate dynamics under limited arbitrage and heterogeneous expectations

Soumya Datta
Assistant Professor (Senior Grade)
Faculty of Economics, South Asian University
Akbar Bhawan, Chanakyapuri
New Delhi 110021, INDIA
Email: soumya@econ.sau.ac.in

Working Paper Number: SAUFE-WP-2017-003

http://www.sau.int/fe-wp/wp003.pdf

FACULTY OF ECONOMICS
SOUTH ASIAN UNIVERSITY
NEW DELHI
August 2017

Exchange rate dynamics under limited arbitrage and heterogeneous expectations

Soumya Datta*

Faculty of Economics, South Asian University, New Delhi, INDIA.
Email: soumya@econ.sau.ac.in

Abstract

This paper attempts to examine the impact of introducing limited arbitrage in a purely deterministic continuous-time model of exchange rates with boundedly rational agents having heterogeneous expectations. The rate of exchange depends on a combination of fundamental factors and speculative behavior by heterogeneous agents in foreign exchange markets. However, given that our focus is primarily on speculators, we keep the determination of fundamentals outside the scope of our model. We find that unlike popular perception in the literature, introduction of limited arbitrage does not necessarily increase the possibilities of deviation from the fundamentals. In fact, under certain situation limited arbitrage might increase the stability of the fundamental equilibrium.

Keywords: Exchange rate, heterogeneous expectations.

JEL classification: C62; D84; E03; F31

*This paper is dedicated to the memory of Prof. Carl Chiarella, who passed away in June 2016, after having a profound impact on research in diverse fields of economics and finance. The current paper is an attempt to re-interpret the model presented in Chiarella, He & Zheng (2013, European Journal of Finance) after introducing limits of arbitrage. Some preliminary results from this paper were presented in 22nd Annual Workshop on Economic Science with Heterogeneous Interacting Agents at Università Cattolica del Sacro Cuore, Milan in June, 2017 and 52nd Annual Meeting of the Indian Econometric Society in January, 2016. The author is grateful to the participants of these conferences for their comments.

1 Introduction

A substantial literature has emerged in recent times, which seeks to explain exchange rate dynamics due to presence of agents with heterogeneous expectations. This rising interest has been partly motivated by certain empirical time series properties of exchange rate dynamics. Traditional models of exchange rate have performed rather poorly in explaining several real life empirical trends. A substantial part of time-series of exchange rates remain unexplained by economical fundamentals, irrespective of the specific theoretical model used to arrive at the these fundamentals [See, for instance, (Lyons 2001)]. In fact, Meese & Rogoff (1983) showed that a random walk model outperforms most standard open economy macroeconomic models in explaining exchange rate dynamics. Real life time-series of exchange rate also shows a substantial volatility, much in excess of fundamental factors, and non-normality, fat tails and power-law behavior which could not be explained by the traditional models of exchange rates which relied on fundamentals alone. This gave rise to a substantial literature on models of exchange rate with boundedly rational agents having heterogeneous expectations. A reasonably exhaustive discussion of this literature might be found, among other places, in de Grauwe & Grimaldi (2006).

One of the early influential models of exchange rate with heterogeneous agents was Frankel & Froot's (1986) attempt to explain the movement of US dollar in the 1980s. This consisted of boundedly rational agents, consisting of fundamentalists who believe in Dornbusch's (1976) overshooting model, chartists or technical analysts who believe that exchange rate follows a random walk, and portfolio managers who form their expectations as a weighted average of the fundamentalists and chartists. They found that the portfolio managers, under linear learning dynamics, learn more slowly about the model than they change it, leading to deviation from the fundamentals and emergence of bubbles in foreign exchange. Chiarella, He & Zheng (2013) modified Frankel & Froot's (1986) model by introducing nonlinearities in the learning dynamics of the portfolio managers and showed several complex dynamical possibilities.

Learning dynamics of the portfolio managers, however, is not the only reason why there might be deviation of the exchange rate from its fundamentals. Shleifer & Vishny (1997) showed, for instance, that as long as the arbitrage is conducted by professional arbitrageurs by pooling resources from outside investors with little knowledge of the market, the ability to conduct arbitrage will be limited by capital constraints arising from agency problems. This might lead to a deviation of the exchange rate from its fundamentals, creating bubbles, leading to a violation of the efficient market hypothesis.

In light of the above literature, the objective of our paper is to examine the effect of limited arbitrage on the speculative activities in foreign exchange market, where the agents are boundedly rational agents and have heterogeneous expectations. We begin with a benchmark model, conisting of a simplified version of Chiarella et al. (2013). Since our focus is on speculative activities, like Westerhoff (2003) and Westerhoff & Reitz (2003) we focus primarily on the speculators. We assume that fundamentals are exogenously given and do not change in the period under consideration. Unlike Chiarella et al. (2013), we do not include nonlinearities in the learning dynamics of portfolio managers. We then introduce limits of arbitrage to examine whether this validates the Efficient Market Hypothesis (EMH).

2 The Model

The model developed in the following sections is similar to the one in Chiarella et al. (2013). We consider a model of exchange rate dynamics where the exchange rate is determined by a combination of fundamental factors and speculative activities by heterogeneous agents. Let $E^\star$ be the fundamental value of the exchange rate, E, (defined as the number of units of domestic currency per unit of foreign currency). Unlike Chiarella et al. (2013), and in line with Westerhoff (2003) and Westerhoff & Reitz (2003), our focus is on the speculative dynamics. Hence, we do not explicitly model the determination of $E^\star$ in our current model – it might either be determined by purchasing power parity, or by interest rate parity condition. If the exchange rate exceeds (falls short of) its fundamental value, i.e. if the domestic currency is undervalued (overvalued), then by increasing (decreasing) the competitiveness of domestic producers, the real part of the economy imparts a downward (upward) pressure on the rate of exchange.

However, the exchange rate also depends on the speculative trading of foreign currency or assets in the financial markets. Following standard literature in this area[1], there exists two categories of traders in the foreign exchange markets:

1. *Fundamentalists*, whose expectations follow a mean-reverting process to the fundamental value, $E^\star$, i.e.
$$\dot{X}_f = \beta_f \left(E^\star - E - X_f\right) \tag{1}$$
where X_f refers to the expected rate of depreciation of the spot exchange rate by the fundamentalists, and $\beta_f \in [0, \infty]$ is the speed of adjustment of the expectations by the fundamentalists; and

2. *Technical analysts* or *chartists*, who believe that the change in the log-exchange rate (LER) follows a declining weighted average of past changes of exchange rate, i.e.
$$\begin{aligned} X_c\left(t\right) &= \beta_c \int_{-\infty}^{t} e^{-\beta_c (t-s)} dE\left(s\right) \\ \Rightarrow \dot{X}_c &= \beta_c \left(\dot{E} - X_c\right) \end{aligned} \tag{2}$$
where X_c refers to the expected rate of depreciation of the spot exchange rate by the technical analysts, and $\beta_c \in [0, \infty]$ is the speed of adjustment of the expectations by the technical analysts.

[1] See, for instance, de Grauwe & Grimaldi (2006)

Let $1-w$ and w be the weightage of fundamentalist and chartist trading practices respectively, where $w \in [0,1]$. We can either interpret $1-w$ and w as the proportion of fundamentalist and chartist traders in the foreign exchange market, or alternately, following Frankel & Froot (1986) and Chiarella et al. (2013), consider a group of portfolio managers who control trading in the foreign exchange markets, and assign a weight of w and $1-w$ to the technical analyst and fundamentalist analysts respectively.

The rate of exchange, as mentioned above, depends on both fundamental factors as well as speculative activities in the financial markets. Let β_s and β_r represent the rates at which the exchange rate responds to speculative activities (weighted average of expectations of both types of agents) and fundamental factors respectively, with $\beta_s, \beta_r \in [0, \infty]$, i.e.

$$\dot{E} = \beta_s \left[(1-w) X_f + w X_c \right] + \beta_r \left(E^\star - E \right) \tag{3}$$

Substituting from (3) into (2), we get

$$\dot{X}_c = \beta_c \left[\beta_s (1-w) X_f - (1 - \beta_s w) X_c + \beta_r \left(E^\star - E \right) \right] \tag{4}$$

The complete dynamical system, therefore, might be described by the following system of differential equations:

$$\dot{X}_f = \beta_f \left(E^\star - E - X_f \right) \tag{5a}$$
$$\dot{X}_c = \beta_c \left[\beta_s (1-w) X_f - (1 - \beta_s w) X_c + \beta_r \left(E^\star - E \right) \right] \tag{5b}$$
$$\dot{E} = \beta_s \left[(1-w) X_f + w X_c \right] + \beta_r \left(E^\star - E \right) \tag{5c}$$

Before we proceed to analyze the complete model, we begin by looking at two special cases: first, where all agents are fundamentalists, i.e. $w = 0$; and second, where all agents are chartists, i.e. $w = 1$.

2.1 Model with only fundamentalist traders

Consider a model with only fundamentalist traders, i.e. $w = 0$. In this case, (5) reduces to the following dynamical system:

$$\dot{X}_f = \beta_f \left(E^\star - E - X_f \right) \tag{6a}$$
$$\dot{E} = \beta_s X_f + \beta_r \left(E^\star - E \right) \tag{6b}$$

The above system represented by (6) has only one steady state, representing the fundamental steady state, given by $(0, E^\star)$. The jacobian computed at the fundamental steady state is given by

$$J = \begin{bmatrix} -\beta_f & -\beta_f \\ \beta_s & \beta_r \end{bmatrix}$$

so that the trace, $-\beta_f - \beta_r < 0$ and determinant, $\beta_f \beta_r + \beta_f \beta_s > 0$, i.e. the fundamental steady state, $(0, E^\star)$ is locally stable from an application of the Routh-Hurwitz

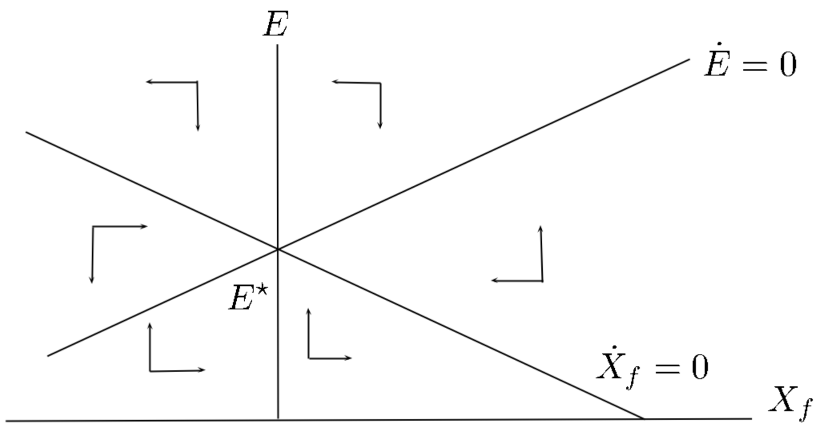


Figure 1: Model with only fundamentalists: phase diagram

condition for stability. In other words, the long-run dynamics will converge to the fundamental steady state, and the Efficient Market Hypothesis holds. The economic intuition is simple: if all agents believe in the fundamental steady state, then their collective trading action will eventually put the economy back to the fundamental steady state.

2.2 Model with only chartist traders

Next, we consider a model with only chartists, i.e. $w = 1$. In this case, (5) is reduced to the following dynamical system:

$$\dot{X}_c = \beta_c\left(\beta_s - 1\right)X_c + \beta_c\beta_r\left(E^\star - E\right) \tag{7a}$$
$$\dot{E} = \beta_s X_c + \beta_r\left(E^\star - E\right) \tag{7b}$$

Once, again, the above system represented by (7) has only one steady state, representing the fundamental steady state, given by $(0, E^\star)$. However, depending on the magnitude of the parameter β_s, representing the sensitivity of the exchange rate to speculative activities by chartist traders, two distinct scenarios emerge. Next, we

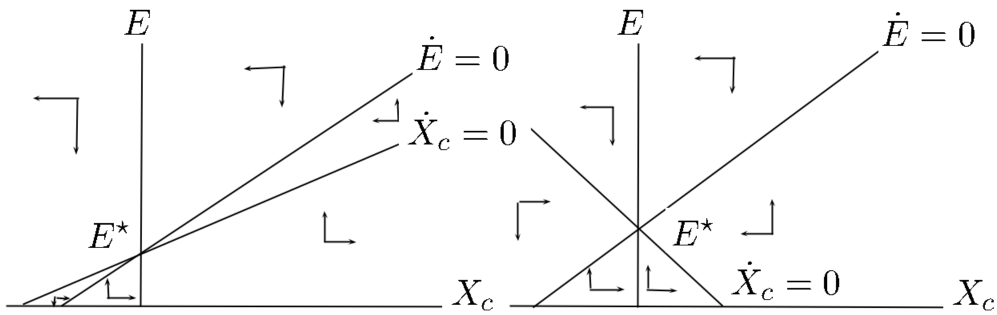


Figure 2: Model with only chartists

check for local stability of the fundamental equilibrium using the Routh-Hurwitz condition. The jacobian computed at the fundamental steady state is given by

$$J = \begin{bmatrix} \beta_c (\beta_s - 1) & -\beta_c \beta_r \\ \beta_s & -\beta_r \end{bmatrix}$$

so that the trace, $\beta_c (\beta_s - 1) - \beta_r \lesseqgtr 0$ for $\beta_s \lesseqgtr 1 + \dfrac{\beta_r}{\beta_c}$. The determinant, $\beta_c \beta_r > 0$. In other words, the fundamental steady state is stable for low values of β_s, the sensitivity of the exchange rate to speculative trading by the chartists. If the trading activities by the chartists are effective enough, the fundamental value becomes unstable. This would be clearer from figure 3. In case 2 (represented by the figure on the right hand side, where $\beta_2 < 1$), the fundamental steady state is always locally stable. In case 1 (represented by the figure on the left hand side, where $\beta_s > 1$), the fundamental steady state is stable only if $\beta_s < 1 + \beta_r/\beta_c$. We also note that for case 1, where $\beta_s < 1$, the fundamental steady state undergoes Andronov-Hopf bifurcation at $\beta_s = 1 + \beta_r/\beta_c$. Subject to the satisfaction of non-degeneracy and tranversality conditions, a unique limit cycle would emerge as β_s passes through critical value. Numerical examples with all the three dynamical possibilities are shown below in figure 3.

2.3 Model with both fundamentalist and chartist traders

Next, we consider the full model represented by (5), consisting of both fundamentalist and chartist traders. We find that the dynamical system represented by (5) will have exactly one steady state given by the fundamental steady state $(0, 0, E^\star)$. The jacobian computed at the fundamental steady state is given by

$$J = \begin{bmatrix} -\beta_f & 0 & -\beta_f \\ \beta_c \beta_s (1 - w) & -\beta_c (1 - \beta_s w) & -\beta_c \beta_r \\ \beta_s (1 - w) & \beta_s w & -\beta_r \end{bmatrix}$$

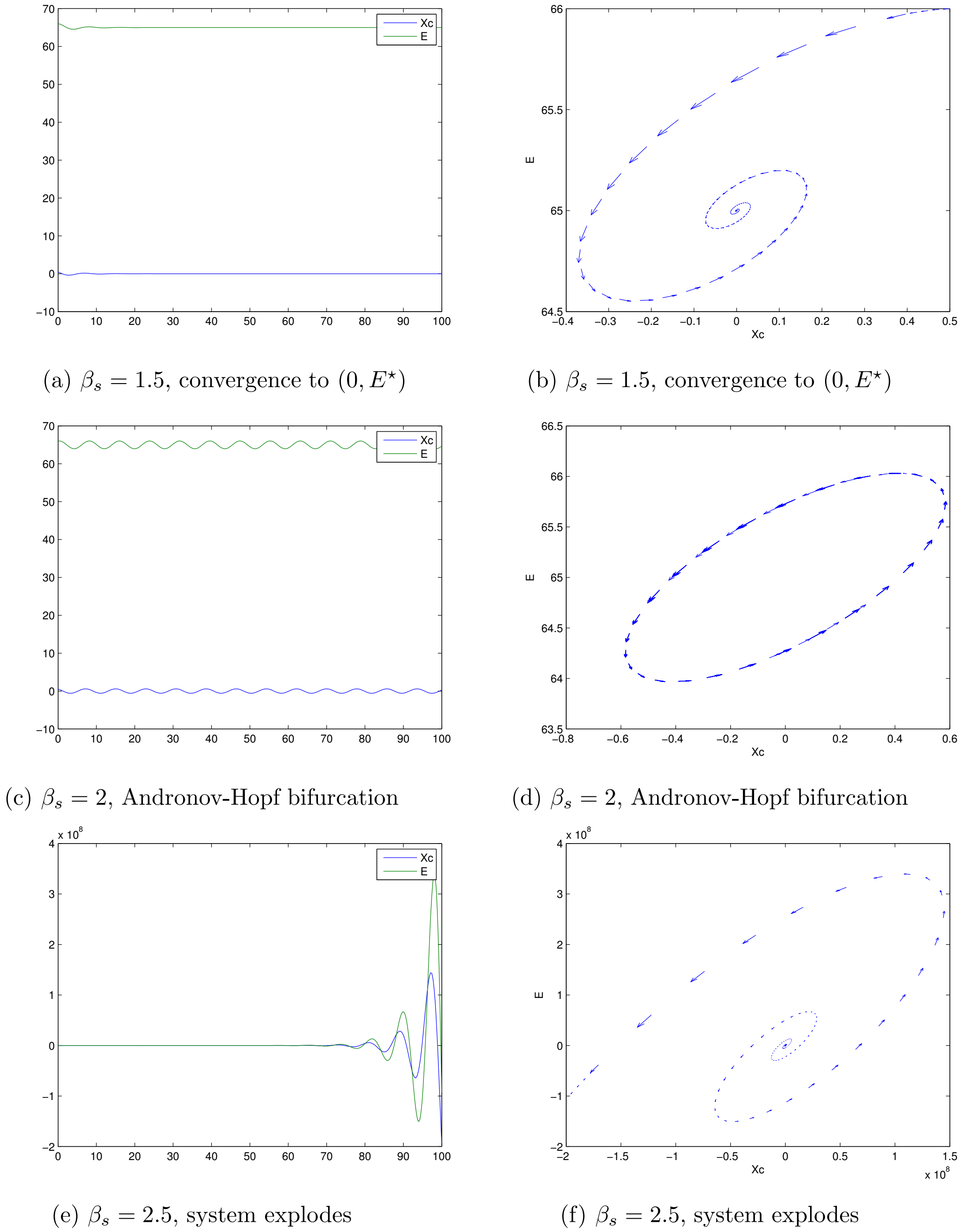
The characteristic equation is given by

$$\lambda^3 + [\beta_c (1 - \beta_s w) + \beta_r + \beta_f] \lambda^2 + [\beta_c \beta_r (1 - \beta_s w) + \beta_c \beta_f (1 - \beta_s w) + \beta_c \beta_r \beta_s w + \beta_f \beta_s (1 - w)] \lambda + \left[\beta_c \beta_f \beta_r (1 - \beta_s w) + \beta_c \beta_f \beta_r \beta_s w + \beta_c \beta_f \beta_s^2 w (1 - w) + \beta_c \beta_f \beta_s (1 - w) (1 - \beta_s w)\right] = 0 \tag{8}$$

To perform stability analysis, we apply the following version of the Routh-Hurwitz condition, found in Flaschel (2009):

> All of the roots of the characteristic equation $\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0$ have negative real parts if and only if the set of inequalities $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 - a_3 > 0$ is satisfied. [cf. Flaschel (2009, page 385, theorem A.5)]

We find that the fundamental steady state is locally stable if the following conditions are satisfied:



(a) $\beta_s = 1.5$, convergence to $(0, E^\star)$

(b) $\beta_s = 1.5$, convergence to $(0, E^\star)$

(c) $\beta_s = 2$, Andronov-Hopf bifurcation

(d) $\beta_s = 2$, Andronov-Hopf bifurcation

(e) $\beta_s = 2.5$, system explodes

(f) $\beta_s = 2.5$, system explodes

Figure 3: Time-series and phase-plots of 100 iterations of numerical solution of the model with only chartists with following parameter values: $\beta_f = 0.8$, $\beta_c = 0.8$, $\beta_r = 0.8$, $E^\star = 65$, initial condition at $(0.5, 66)$. β_s is used as bifurcation parameter.

$$\begin{aligned} a_1 &= \beta_c (1 - \beta_s w) + \beta_r + \beta_f > 0 \\ \Leftrightarrow w &< \frac{1}{\beta_s} + \frac{\beta_r + \beta_f}{\beta_s \beta_c} \end{aligned} \tag{9}$$

$$\begin{aligned}
a_3 &= \left[\beta_c\beta_f\beta_r\left(1-\beta_s w\right)+\beta_c\beta_f\beta_r\beta_s w+\beta_c\beta_f\beta_s^2 w\left(1-w\right)+\right.\\
&\left.\beta_c\beta_f\beta_s\left(1-w\right)\left(1-\beta_s w\right)\right]>0\\
&\Leftrightarrow \beta_c\beta_f\beta_r+\beta_c\beta_f\beta_s\left(1-w\right)>0 \text{ which holds true since } w<1 \qquad (10)
\end{aligned}$$

$$\begin{aligned}
a_1a_2-a_3 &= \left[\beta_c\left(1-\beta_s w\right)+\beta_r+\beta_f\right]\left[\beta_c\beta_r\left(1-\beta_s w\right)+\beta_c\beta_f\left(1-\beta_s w\right)+\right.\\
&\left.\beta_c\beta_r\beta_s w+\beta_f\beta_s\left(1-w\right)\right]-\left[\beta_c\beta_f\beta_r\left(1-\beta_s w\right)+\beta_c\beta_f\beta_r\beta_s w+\right.\\
&\left.\beta_c\beta_f\beta_s^2 w\left(1-w\right)+\beta_c\beta_f\beta_s\left(1-w\right)\left(1-\beta_s w\right)\right]>0\\
&\Leftrightarrow w<\bar{w}_1 \text{ or } w>\bar{w}_2 \qquad (11)
\end{aligned}$$

where $\bar{w}_1$ and $\bar{w}_2$ are the roots of the quadratic equation $a_1a_2 - a_3 = 0$, expressed in terms of w. From (9), (10) and (11), we have $w < \underline{\mathrm{w}}$ as a *sufficient* condition[2] for local stability of the fundamental steady state, where

$$\underline{\mathrm{w}} = \min\left[\frac{1}{\beta_s}+\frac{\beta_r+\beta_f}{\beta_s\beta_c}, \bar{w}_1, 1\right]$$

In other words, the fundamental steady state is stable, as long as the proportion of speculators deviating from fundamentalist or rational expectations does not exceed a limit. As long as there are enough agents holding fundamentalist expectations, fundamental steady state is stable.

We also note that $\underline{\mathrm{w}}$ depends inversely on β_s. In other words, more the influence of speculative trading activities on the exchange rates, less is the room for the proportion of agents allowed by the system to deviate from rational expectations without destabilizing the system.

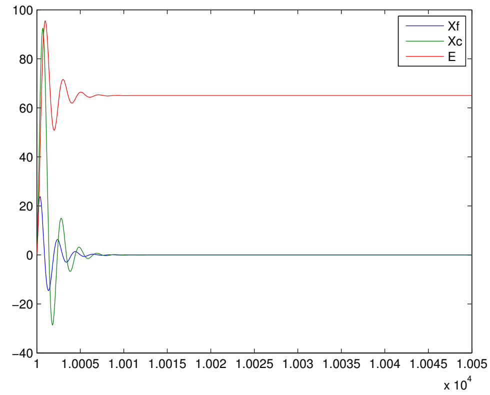
A number of possibilities emerge in case the fundamental steady state gets destabilized. Note that

$$\begin{aligned}
a_2 &= \left[\beta_c\beta_r\left(1-\beta_s w\right)+\beta_c\beta_f\left(1-\beta_s w\right)+\beta_c\beta_r\beta_s w+\beta_f\beta_s\left(1-w\right)\right]\\
&\Leftrightarrow w<\frac{1}{1+\beta_c}+\frac{\beta_c\beta_r+\beta_c\beta_f}{\beta_s\beta_f\left(1+\beta_c\right)} \qquad (12)
\end{aligned}$$

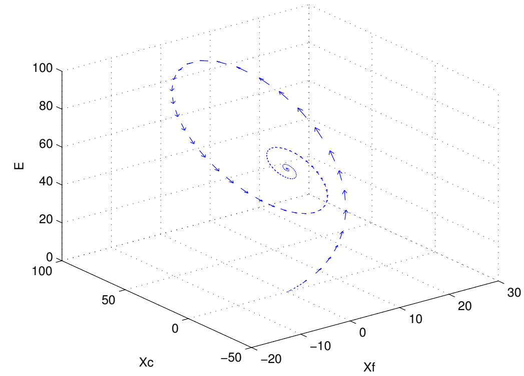
Therefore, the fundamental steady state undergoes a Hopf bifurcation as the control parameter, w, representing the proportion of chartists, passes through $\underline{\mathrm{w}}$. Subject to the non-degeneracy and transversality conditions being satisfied, a unique limit cycle emerges at this point.

Numerical examples of the above possibilities are shown below in figure 4.

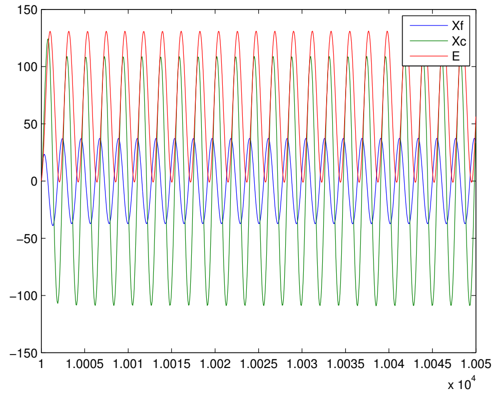
[2]The *sufficient* condition will also be *necessary* condition, provided $\bar{w}_2 \geq \min\left[\frac{1}{\beta_s}+\frac{\beta_r+\beta_f}{\beta_s\beta_c}, 1\right]$



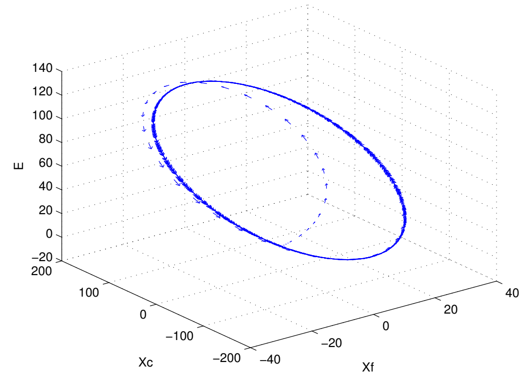
(a) $w = 0.1$, convergence to $(0, 0, E^\star)$



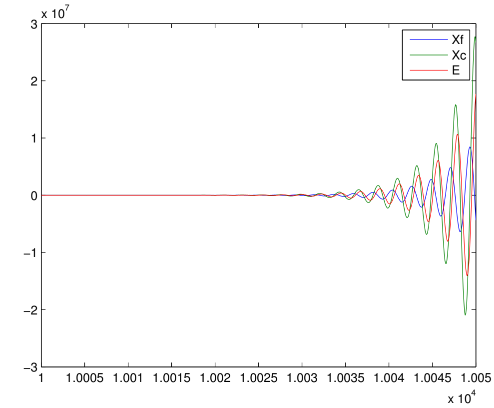
(b) $\beta_s = 1.5$, convergence to $(0, 0, E^\star)$



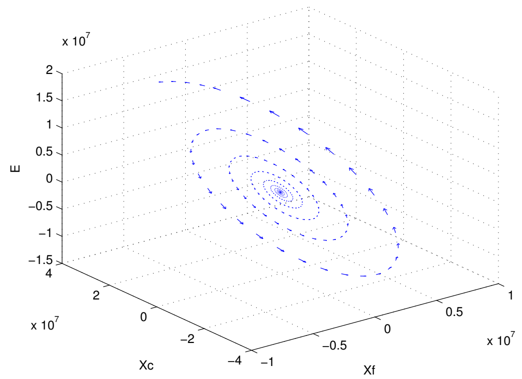
(c) $w = 0.25$, Hopf bifurcation



(d) $w = 0.25$, Hopf bifurcation, limit cycle



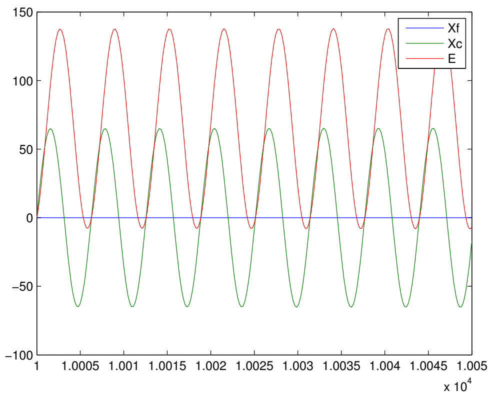
(e) $w = 0.3$, system explodes



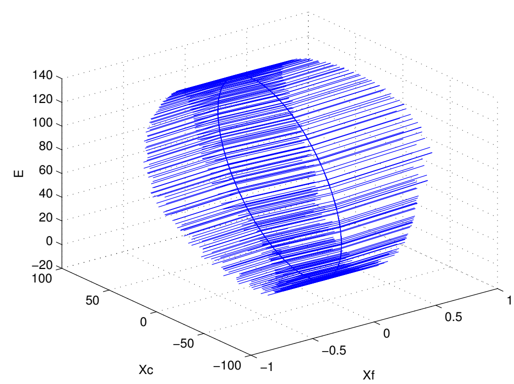
(f) $w = 0.3$, system explodes

Figure 4: Time-series and phase-plots of 10000 to 10050 iterations of numerical solution of the model with both fundamentalists and chartists, with following parameter values: $\beta_f = 2$, $\beta_c = 2$, $\beta_r = 0.5$, $\beta_s = 5$, $E^\star = 65$, initial condition at $(0, 0, 0)$. w is used as bifurcation parameter.

Finally, if either of the two classes of agents, fundamentalists and chartists, fail to adjust their expectations according to their respective rules of adjustment, so that either β_c or β_f (but not both) are zero, then we have an interesting situation where $a_3 = 0$, while $a_1, a_2 > 0$. This would represent an emergence of *fold* or *saddle-node* bifurcation. The literature on bifurcation theory suggests that this might open up several dynamical possibilities, including generation of multiple homoclinic orbits, disappearance of saddle-nodes through *Shil'nikov* bifurcation leading to complex dynamics due to generation of an infinite number of saddle-periodic orbits. A detailed discussion of these possibilities might be found in Kuznetsov (1997, chapt. 3 & 6). A simple numerical example of such complicated behavior is provided below in figure 5.



(a) Time-series of saddle-node bifurcation



(b) Phase diagram of saddle-node bifurcation

Figure 5: Saddle node bifurcation occurs when $\beta_c = 0$, i.e. chartists fail to adjust.

3 Limited arbitrage in foreign exchange markets

In the previous sections, we have assumed the proportion of fundamentalist and chartist traders (or alternately, weights assigned by the portfolio managers to these two forms of analysis), w, to be exogenously given. In this section, we attempt to endogenize w. For this purpose, we consider a simple model of limits to arbitrage in the lines of arguments found in Shleifer & Vishny (1997).

Shleifer & Vishny (1997) points out that the model of arbitrage implicit in standard models like the one found in Fama (1970) assumes a market with very large number of tiny arbitrageurs, each taking a position against mispricing against variety of markets. Since each position is small, capital constraints are not binding and each arbitrageur is effectively risk-neutral toward each trade. Their collective actions drive prices towards fundamental values. However, in a market where arbitrage is conducted by relatively few professional investors (both fundamentalists and chartists) who pool in resources from outside investors with little knowledge of the markets (either in the form of borrowing or investments in portfolios, say, in the form of mutual funds), standard agency problems might arise. The ability of the professional traders to conduct arbitrage in this case will be limited by capital constraints. This will be particularly evident if the markets are grossly mispriced, leading to the fundamentalist trading position consistently resulting in losses. In a world without capital constraints, eventually the fundamentalists would recover their losses once the prices come back to fundamentals; however, capital constrainsts might force fundamentalists to switch their positions much before this happens, resulting in the markets continuing to be mispriced.

We should note that the above argument goes against the interpretation of limits to arbitrage made by de Grauwe & Grimaldi (2006). They argued that a greater deviation of prices from their fundamentals would make the fundamentalist trading strategy less risky. However, this argument is based not on limits of arbitrage due to agency constraints per se, but on the lack of accuracy of the speculators in measuring the true value of the fundamentals. In other words, the argument is primarily based on imperfect information. We, on the other hand, argue that borrowing constraints make the fundamentalist trading strategy more risky during periods of larger deviation from fundamentals, since the uninformed outside investors (either lenders or mutual fund investors) face large short-run losses. In this sense our formulation is closer to Shleifer & Vishny (1997) than the one posed by de Grauwe & Grimaldi (2006).

In light of above arguments, let us model the ratio of fundamenlists to chartists in the following manner:

$$\frac{w}{1-w} = a\,(E - E^{\star})^2 \tag{13}$$

In other words, this ratio is a quadratic function of the absolute deviation of the exchange rates from their fundamentals.

We should note that the above formulation makes a fundamental distinction between the two kinds of agents in the market. The chartists trading strategy is based on realized profits, and hence, face a less severe financial constraint compared to the fundamentalists whose ability to arbitrage depend on their ability to finance their trading strategy.

From (13), we have

$$w = \frac{a\,(E^{\star} - E)^2}{1 + a\,(E^{\star} - E)^2}, \quad \text{and} \tag{14}$$

$$1 - w = \frac{1}{1 + a\,(E^{\star} - E)^2} \tag{15}$$

representing the proportion of chartists and fundamentalists (or alternately, the weightage assigned by the portfolio manager to technical analysis and fundamentalism) respectively. Note that at its fundamental value, $E = E^{\star} \;\Leftrightarrow\; w = 0$, i.e. the chartists disappear when the economy attains the fundamental steady state.

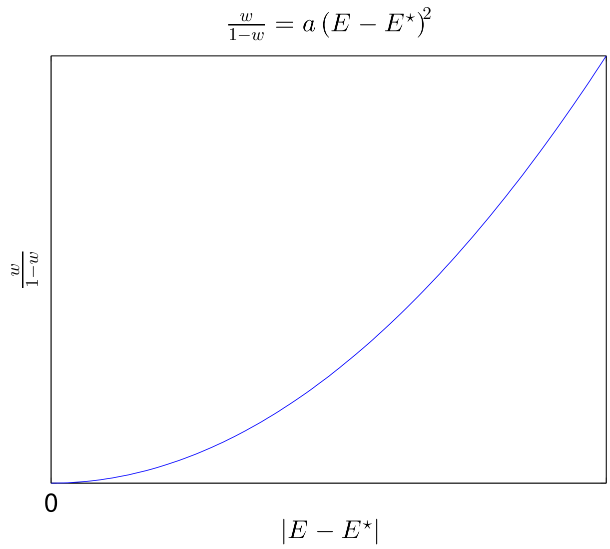


Figure 6: Limits to arbitrage

Next, we endogenize the proportion of technical analysts in the market using the above relation. Substituting for w in (5), we get the following dynamical system:

$$\dot{X}_f = \beta_f \left(E^\star - E - X_f\right) \tag{16a}$$

$$\dot{X}_c = \beta_c \left[\frac{\beta_s}{1 + a\left(E^\star - E\right)^2} X_f - \frac{1 + a\left(1 - \beta_s\right)\left(E^\star - E\right)^2}{1 + a\left(E^\star - E\right)^2} X_c + \beta_r \left(E^\star - E\right)\right] \tag{16b}$$

$$\dot{E} = \frac{\beta_s}{1 + a\left(E^\star - E\right)^2} \left[X_f + a\left(E^\star - E\right)^2 X_c\right] + \beta_r \left(E^\star - E\right) \tag{16c}$$

The dynamical system represented by (16) will have *four* steady states, one of which is the fundamental steady state represented by $(0, 0, E^\star)$.

We first examine the fundamental steady state, $(0, 0, E^\star)$. Recall that at this steady state, $w = 0$, i.e. the chartists completely disappear from the system at the fundamental steady state. The jacobian computed at the fundamental steady state is as follows:

$$J = \begin{bmatrix} -\beta_f & 0 & -\beta_f \\ \beta_c \beta_s & -\beta_c & -\beta_c \beta_r \\ \beta_s & 0 & -\beta_r \end{bmatrix}$$

The characteristic equation is given by

$$\lambda^3 + \left(\beta_c + \beta_f + \beta_r\right)\lambda^2 + \left(\beta_c \beta_f + 2\beta_c \beta_r + \beta_s\right)\lambda + \left(\beta_c^2 \beta_r + \beta_c \beta_s\right) = 0 \tag{17}$$

Applying Routh-Hurwitz criterion for stability, we find that the fundamental steady state is always locally stable. Comparing with the benchmark model, we find that introduction of limited arbitrage seems to increase the stability of the fundamental equilibrium. In other words, unlike what Shleifer & Vishny (1997) expected, introduction of limited arbitrage to our benchmark model seems to strengthen efficiency market hypothesis.

Even though this result is contrary to much of the literature on limits of arbitrage, it is not altogether surprising. Note that given the way we modeled limits of arbitrage, chartists keep switching to fundamentalism as the exchange rate moves closer to the fundamental steady state. This creates a tendency for exchange rates to converge to the fundamental steady state, where all chartists have already switched to fundamentalism.

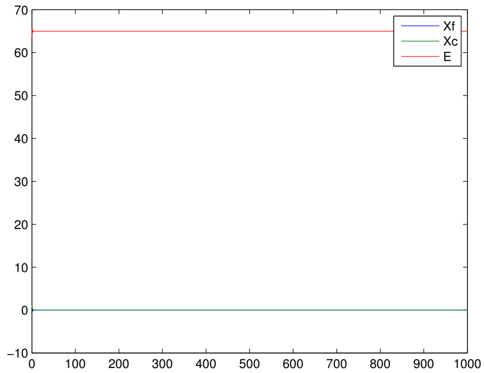
However, we should note that unlike the benchmark model, we now also have four other non-fundamental steady state. An application of Routh-Hurwitz condition shows that these non-fundamental steady states are conditionally stable. The actual dynamics, therefore would depend on initial conditions. Again, this is not surprising. As we move away from fundamental steady state, fundamentalists face capital constraints and are forced to switch to technical analysis. This creates condition for further deviation from the fundamentals.

The above point will be clearer from the following numerical example in figure 7. In figure 7a and 7b, starting from an initial point sufficiently close to the fundamental steady state, the solution converges to the steady state. However, in figures 7c and 7d, starting from an initial point slightly further from the fundamental steady state, the solution does not converge to any state and instead explodes.

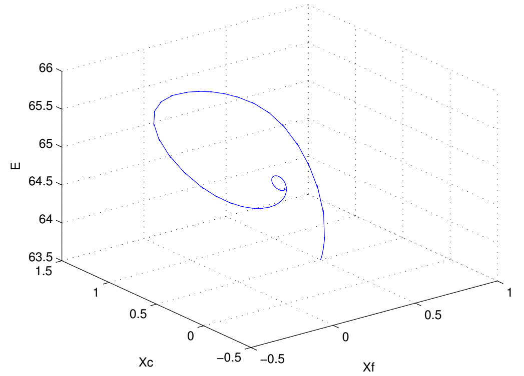
In other words, as long as one is adequately close to the fundamental steady state, the solution converges to it. However, for any perturbation which is not close enough to the fundamental steady state, the solution explodes (like a bubble) and does not converge to the steady state. The economic intuition behind this should be quite clear: for small perturbations away from the steady state, the fundamentalist arbitrageurs are able to conduct arbitrage to bring back the solution to the fundamental equilibrium; however, for larger perturbations, their ability to arbitrage is limited by their ability to finance the arbitrage which becomes increasingly difficult due to agency problem.

4 Concluding Remarks

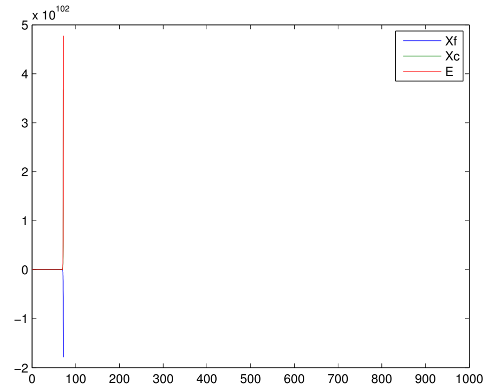
We considered a series of models in this paper, where the exchange rate depends on a combination of fundamental and speculative factors. The first set of models have the proportion of fundamentalists and chartists given exogenously. We find that in the presence of only fundamentalists, the solution converges to the steady state, whereas in the presence of only technical analysts or chartists, the solution might either converge to the steady state, or diverge away from it or move around it in the form of a limit cycle. The exact outcome will depend on the speed of adjustment of the chartist speculators.



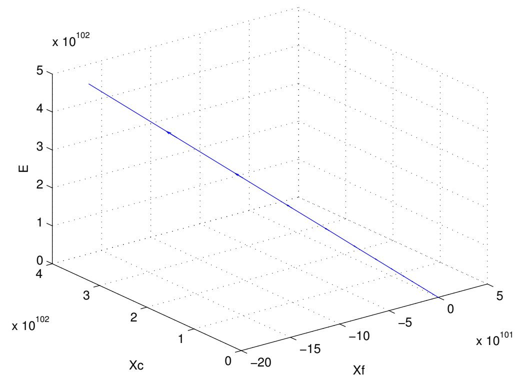
(a) Time-series with $(0.5, 0.5, 63.5)$ as initial point



(b) Phase diagram with $(0.5, 0.5, 63.5)$ as initial point



(c) Time-series with $(0.5, 0.5, 63)$ as initial point



(d) Phase diagram with $(0.5, 0.5, 63)$ as initial point

Figure 7: Numerical solution to (16) from alternative initial points, with following set of parameter values: $\beta_f = 2$, $\beta_c = 0.5$, $\beta_r = 0.5$, $\beta_s = 5$, $E^\star = 65$, $a = 0.5$. Steady states are given by $(0, 0, 65)$ or the fundamental equilibrium, and $(-4.9174, 0.9391, 69.9174)$, $(4.9174, -0.9391, 60.0826)$, $(-1.3489, 1.7658, 66.3489)$ and $(1.3489, -1.7658, 63.6511)$.

In the presence of both fundamentalist and chartist speculators, the outcome depends on the proportion of chartist speculators. As the proportion of chartist speculators cross a threshold limit, the steady state changes from stable to unstable via Hopf bifurcation, leading to emergence of limit cycles. In other words, presence of a large number of chartists exceeds a certain threshold, they are able to pull an economy away feom fundamental equilibrium. There are also possibilities of more complicated dynamics emerging from this setup.

The most important contribution of this paper, however, as been a re-interpretation of limits of arbitrage, of Shleifer & Vishny (1997) within the heterogeneous agent model. We find that introduction of limits of arbitrage, in some sense, has a stabilizing impact on the fundamental equilibrium. This is because, for small perturbations from the fundamental equilibrium, the switching of agents from fundamentalism to chartism ensure a quick return to the fundamentals. For larger perturbations, however, limits of arbitrage starts getting stronger, leading to fundamentalist speculators switching to chartism, leading to the solution not returning to its fundamentals.

We must, however, point out that a complete understanding of the nature of the dynamics would require a more detailed look at the non-fundamental steady states, so that we can precisely point out exactly when does the solution stops converging the fundamental steady state. A more realistic model should also incorporate some stochastic noise into the model to analyze the how soon the system reverts to the mean. Introduction of such noise will also provide us with a more realistic time-series. We leave these these as possible areas for future research.

References

Chiarella, C., He, X.-Z. & Zheng, M. (2013), 'Heterogeneous expectations and exchange rate dynamics', *The European Journal of Finance* **19**(5), 392–419.

de Grauwe, P. & Grimaldi, M. (2006), *The exchange rate in a behavioral finance framework*, Princeton University Press, New Jersey.

Dornbusch, R. (1976), 'Expectations and exchange rate dynamics', *Journal of Political Economy* **84**(6), 1161–1176.

Fama, E. F. (1970), 'Efficient capital markets: A review of theory and empirical work', *The Journal of Finance* **25**(2), 383–417.

Flaschel, P. (2009), *The Macrodynamics of Capitalism: Elements for a Synthesis of Marx, Keynes and Schumpeter*, second revised and enlarged edn, Springer Verlag, Berlin Heidelberg.

Frankel, J. & Froot, K. (1986), 'Understanding the US dollar in the eighties: The expectations of chartists and fundamentalists', *The Economic Record* **Special issue**, 24–38.

Kuznetsov, Y. A. (1997), *Elements of Applied Bifurcation Theory*, Vol. 112 of *Applied Mathematical Sciences*, second edn, Springer-Verlag, New York.

Lyons, R. (2001), *The microstructure approach to exchange rates*, Applied Mathematical Sciences, MIT Press, Cambridge.

Meese, R. & Rogoff, K. (1983), 'Empirical exchange rate models of the seventies: Do they fit out of sample?', *Journal of International Economics* **14**(1-2), 3–24.

Shleifer, A. & Vishny, R. W. (1997), 'The limits of arbitrage', *The Journal of Finance* **LII**(1), 35–55.

Westerhoff, F. (2003), 'Expectations driven distortions in foreign exchange market', *Journal of Economic Behavior and Organization* **51**(3), 389–412.

Westerhoff, F. & Reitz, S. (2003), 'Nonlinearities and cyclical behavior: The role of chartists and fundamentalists', *Studies in Nonlinear Dynamics and Econometrics* **7**(4).